# Discontinuous solutions of continuous time general equilibrium models

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**Economic models**

Is economic time discrete or continuous?

| | |
|---|---|
| Discrete time | Arrow-Debreu model [2], [3], [6]<br>Overlapping-generations model [4], [9] |
| Continuous time | Neoclassical and endogenous growth models |

Continuous time economic models are better than discrete time models at representing qualitatively **discontinuous** phenomena.

The long history of the study of discontinuous phenomena in continuous (or differentiable) models.

Catastrophe theory and the theory of singularities of smooth mappings: [11], [1].

In Economics, study of the Arrow-Debreu model with the tools of differential topology: [3].

## Fundamentals of the Arrow-Debreu model

$\ell$ goods
commodity space $\mathbb{R}^\ell$
consumption space

$X = \mathbb{R}^\ell_+ = \{x = (x^1, x^2, \ldots, x^\ell) \in \mathbb{R}^\ell \mid x^j \geq 0 \text{ for every } j\ \}$

## Consumers

utility function $u_i : X \to \mathbb{R}$
price vector $p \in \mathbb{R}^\ell_{++}$
Prices are often normalized by the numeraire convention $p_\ell = 1$

**Consumer $i$'s maximization problem:**

$$\text{Maximize} \quad u_i(x_i)$$

subject to

$$p \cdot x_i \leq w_i \quad \text{and} \quad x_i \in X$$

Under suitable assumptions: a solution exists, and is unique: $f_i(p, w_i)$.

**Smooth Arrow-Debreu model:**

1) $u_i$ is $C^\infty$;

2) $Du_i(x_i) \in X$;

3) Let $X \in \mathbb{R}^\ell$:

$$\left.\begin{array}{c} X^T D^2 u_i(x_i) X \geq 0 \\ Du_i(x_i)^T X = 0 \end{array}\right\} \Rightarrow X = 0.$$

4) $u_i^{-1}(c)$ is closed in $\mathbb{R}^\ell$ for all $c \in \mathbb{R}$.

The demand function $f_i : S \times \mathbb{R}_{++} \rightarrow \mathbb{R}^{\ell}$ is smooth satisfies Walras law (i.e., $p \cdot f_i(p, w_i) = w_i$ for all $p$ and $w_i$)

satisfies the Slutsky property (symmetry and negative definiteness of the Slutsky matrix associated with the demand function),

is a diffeomorphism.

See [3].

**Equilibrium**

Distribution of individual endowment:
$\omega = (\omega_1, \omega_2, \ldots, \omega_m) \in X^m$

Space of individual endowments.
$\Omega = X^m$.

**Definition**
The pair $(p, \omega) \in S \times \Omega$ is an equilibrium if

$$\sum_i f_i(p, p \cdot \omega_i) = \sum_i \omega_i.$$

## Discrete time dynamic version of the Arrow-Debreu model

Goods are dated!
Time periods $t = 0, 1, \ldots, T$
$\ell$ goods per time period.
Commodity space $\mathbb{R}^{\ell(T+1)}$
Consumption space $\mathbb{R}^{\ell(T+1)}_{++}$
Utility function:

$$U_i(x_i) = \sum_{t=0}^{T} \delta_i^t u_i(x_i(t))$$

## consumer's $i$ maximization problem

$$\text{Maximize} \quad U_i(x_i)$$

subject to

$$p \cdot x_i = \sum_{t=0}^{T} p(t) \cdot x_i(t) = w_i.$$

**Equilibrium**

Consumer $i'$s endowments:
$\omega_i = (\omega_i(t))$ for $t = 0, 1, \ldots, T$

Distribution of individual endowments:
$\omega = (\omega_i)$.

**Definition**

The pair $(p, \omega)$ is an equilibrium if

$$\sum_i f_i(p, p \cdot \omega_i) = \sum_i \omega_i.$$

## Continuous time version of the Arrow-Debreu model

Time interval: $[0, 1]$
$\ell$ goods at each date $t \in [0, 1]$
Commodity space: some subset of the set of maps from $[0, 1]$ into $\mathbb{R}^\ell$
Consumption space: some subset of the set of maps from $[0, 1]$ into $\mathbb{R}^\ell_{++}$.
Price functions: $p : [0, 1] \to \mathbb{R}^\ell_{++}$.
Normalization $p_\ell(0) = 1$.

## Utility function

$$U_i(x_i) = \int_0^1 u_i(t, x_i(t))dt$$

## budget constraint

$$\int_0^1 p(t) \cdot x_i(t)dt \leq w_i.$$

**Observations**

It follows from Tonelli's semi-continuity theorem (See for example [5]) that the concavity of $u_i(t, .)$ is essentially equivalent to the upper semicontinuity of the functional

$$\int_0^1 u_i(t, x_i(t))dt$$

with respect to $x_i \in H^{1,s}((0,1), \mathbb{R}^\ell)$ for $s \geq 1$ (Sobolev space).

Note also that this is typically an isoperimetric problem. The difference with the standard formulations is that the integrand $p(t) \cdot (x_i(t) - \omega_i(t))$ is not necessarily continuous with respect to $t \in [0, 1]$.

**The space of functions of bounded variations**

Introduced by Camille Jordan.

Let $f : [0, 1] \to \mathbb{R}$ not necessarily continuous.

*Total variation function*

$$x \in (0,1) \quad T_f(x) = \sup \sum_{j=1}^{N} |f(x_j) - f(x_{j-1})|$$

over all integers $N$ and all choices of $\{x_1, x_2, \dots, x_N\}$ such that

$$0 < x_1 < x_2 < \cdots < x_N < x.$$

$$x < y \quad \Rightarrow 0 \leq T_f(x) \leq T_f(y) \leq \infty$$

**Definition**

$$V_0^1(f) = \lim_{x \to 1-0} T_f(x).$$

If $V_0^1(f) < \infty$, then $f$ is a *function of bounded variation.*

$BV(0,1)$ the space of functions of bounded variation on $(0,1)$.

**Proposition**

Let $f \in BV(0,1)$. Then $f(x-0)$ exists for $x \in (0,1]$ and $f(x+0)$ for $x \in [0,1)$. The set of points at which $f$ is discontinuous is at most countable.

There is a unique constant $c$ and a unique function with bounded variation, which is left-continuous and satisfies

$$\lim_{x \to a+0} g(x) = 0$$

so that

$$f(x) = c + g(x)$$

at all points of continuity of $f$.
Functions of bounded variation equal to $c + g(x)$ are said to be normalized. Let $NBV(0,1)$ the set of normalized bounded variation functions.

**Lemma**
Let $f \in BV(0,1)$ and $x < y$.
Then,

$$|f(x) - f(y)| \leq T_f(y) - T_f(x)$$

$\varepsilon > 0$, there exists $0 < x_1 < x_2 < \cdots < x_N = x$ such that

$$\sum_{i=1}^{N} |f(x_i) - f(x_{i-1})| > T_f(x) - \varepsilon$$

$$T_f(y) \geq |f(y) - f(x)| + \sum_{i=1}^{N} |f(x_i) - f(x_{i-1})|$$

$$T_f(y) > |f(x) - f(y)| + T_f(x) - \varepsilon$$

**Consequences**

1) The sequence $(f(x_n))$ is Cauchy if the sequence $(T_f(x_n))$ is also Cauchy.

2) $T_f$ monotone $\Rightarrow$ right and left limits at every point, and at most a countable number of discontinuities $\Rightarrow$ same properties for $f$.

3) $c = \lim_{t\to 0+0} f(t)$ and $g(x) = f(x-0) - c$;
Then $g(x)$ is left-continuous and $V_0^1(g) \leq V_0^1(f)$.

4) If $f \in BV(0,1)$, then $f$ is differentiable almost everywhere and $[f'] \in L^1(0,1)$.

5) The space $BV(a,b)$ is not separable.

6) For every bounded sequence in $BV(0,1)$, there exists a subsequence converging almost everywhere on $(0,1)$.

7) For every $p < \infty$, the embedding $BV(0,1) \to L^p(0,1)$ is compact.

8) The convergence on $BV(0,1)$ (called the $BV-$ weak* convergence) defined by

$$\begin{cases} u_n \to u & \text{strongly in } L^1(a,b) \\ u_n' \to u' & \text{weakly* in } M(0,1) \end{cases}$$

is such that norm bounded sequences in $BV(0,1)$ are $BV$-weakly* compact.

For more details about functions of bounded variation, see [5] and [10]

**Back to the consumer's maximization problem**
Find $x_i \in NBV(0,1)^\ell$ solution of:

$$\text{Maximize} \quad \int_0^1 u_i(t, x_i(t))dt$$

subject to

$$\int_0^1 p(t) \cdot (x_i(t) - \omega_i(t))dt \leq 0$$

where $\omega_i \in NBV(0,1)^\ell$ and $p \in NBV(0,1)^\ell$.

Let $u_i : [0,1] \times X \to \mathbb{R}$ be $C^k$, such that $u_i(t, .)$ satisfies the assumptions (1) to (4), with $k \geq 0$.

**Proposition** This problem has a solution in $NBV(0,1)^\ell$ for any $p \in NBV(0,1)^\ell$.

This solution is denoted by

$$x_i(p, w_i)$$

where $w_i = \int_0^1 p(t) \cdot \omega_i(t)dt$.

## Equilibrium in the continuous time setup

**Definition.** The price function $p \in NBV(0,1)^\ell$ is an equilibrium price function if

$$\sum_i x_i(p, \int_0^1 p(t) \cdot x_i(t)dt)(\theta) = \sum_i \omega_i(\theta)$$

for all $\theta \in [0,1]$.

**Theorem.** If the function $r = \sum_i \omega_i$ is $C^p$, with $0 \leq p \leq k$, then every equilibrium price function associated with $\omega = (\omega_i)$ is $C^p$.

*Sketch of the proof.*
Step 1. Let $x = (x_i)$ be the equilibrium allocation associated with the equilibrium price function $p \in NBV(0,1)^\ell$. Then $x = (x_i)$ is a Pareto optimum.
Step 2. If $x = (x_i)$ is a Pareto optimum, there exists a set of multipliers $(1, \lambda_2, \ldots, \lambda_m) \in \{1\} \times \mathbb{R}^{m-1}_{++}$ such that $x$ is the solution of the problem of maximizing

$$\mathcal{L}(x, \lambda) = U_1(x_1) + \lambda_2 U_2(x_2) + \cdots + \lambda_m U_m(x_m)$$

subject to

$$x_1 + x_2 + \cdots + x_m = r.$$

Step 3. The allocation $x = (x_i)$ is a solution of the above maximization problem if and only if $x(t)$ is a solution of the problem of maximizing

$$u_1(t, x_1(t)) + \lambda_2 u_2(t, x_2(t)) + \cdots + \lambda_m u_m(t, x_m(t))$$

subject to the constraint

$$x_1(t) + x_2(t) + \cdots + x_m(t) = r(t).$$

Step 4. The component $x(t) = (x_i(t))$ is a smooth function of $r(t)$ and $\lambda_2, \ldots, \lambda_m$, and $C^k$ of $t \in (0,1)$.
Step 5. The first order necessary conditions take the form

$$\lambda_i D u_i(t, x_i(t)) = (\mu^1(t), \ldots, \mu^\ell(t))$$

Hence $\mu(t) \in X$ is $C^k$.
Step 6. The price function $p$ is of the form $p = \alpha\mu$, where $\alpha > 0$ is constant. Therefore, $p$ is $C^k$.

**Equilibrium with liquidity constraints: The polar case**
We have the fully constrained liquidity problem if the budget constraints of consumer $i$ take the form

$$p(t) \cdot (x_i(t) - \omega_i(t)) \leq 0$$

for every $t \in [0, 1]$ and every $i$.
**proposition** The solution of the consumer′s maximization problem satisfies

$$\text{Maximize} \quad u_i(x_i(t))$$

subject to

$$p(t) \cdot (x_i(t) - \omega_i(t)) = 0.$$

**corollary**
The allocation $x$ is an equilibrium of the of the fully constrained liquidity continuous time model if and only if $x(t)$ is an equilibrium allocation of the economy defined by the utility functions $u_i(t, x_i(t))$ and the resources $\omega_i(t)$ with $i = 1, 2, \ldots, m$.

**corollary**
There are economies for which no continuous equilibrium solutions exist even with $C^\infty$ endowments.

**liquidity constraints**

Weaker liquidity constraints can take various forms. One that corresponds to every day practice is the following

$$\int_0^t p(\theta) \cdot (x_i(\theta) - \omega_i(\theta))d\theta \leq \alpha \int_t^1 p(\theta) \cdot \omega_i(\theta)d\theta$$

with $\alpha > 0$. In practice, often $\alpha$ is somewhere in between $0$ and $.2$. One could extend this model easily by making $\alpha$ a function of time.

One observes that for $\alpha = 0$, one gets the fully liquidity constrained model. For $\alpha$ large enough, one gets the standard Arrow-Debreu (with just one budget constraint per consumer).

Therefore, one can expect that the introduction of liquidity constraints (by the imposition of, for example, some restrictive monetary policy) may induce discontinuities of the solutions...

## Conclusion

Relationship with Hilbert's XIXth problem on the regularity of elliptic partial differential equations.

For a survey of related issues in the Calculus of Variations, see [7], [8], [5].

The regularity problem in the continuous time Arrow-Debreu model is easier to handle than in the Calculus of Variations. Nevertheless, it is very likely that extending the study to incomplete financial markets (and therefore to stronger liquidity constraints than in the Arrow-Debreu model) will require some of the techniques and methods of the calculus of variations, not the least being the tools and methods of the direct approach with which the Italian mathematical community has been associated with since its early beginnings.

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